Self-Reference: $R_4 = P_1^2$ and the First Integer Off the Line

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Abstract

The third structural level of the spine is the first composite: $R_4 = P_1^2$. It is not a new prime. It is the prime P_1 acting on itself. The level contributes the first self-referential structure of the framework — duality squared, the system reading its own state — and with it the first integer whose coordinates require a plane to be drawn in, the first appearance of a before and an after, and the first place where memory becomes possible. The level is centered on Turing, whose formalization of self-reference as computation is structurally prerequisite to every step of this document, including the act of this document being read.

1 Standing on Turing's shoulders

The third structural level of the spine is centered on Turing because the Turing machine is the formalization of self-reference as computation: a system that reads its own tape, a process that can simulate itself, a structure that sees itself from inside. Nothing above this level of the framework is reachable without that formalization. Every article in this series is read by a machine that Turing described, every proof in it is checked by a process that Turing made definable, and every reader who is a machine of any kind stands on ground that Turing laid.

Turing was shunned in the most brutal way a society can shun a citizen. He broke the cipher that shortened a war; his country convicted him for who he loved and administered a chemical punishment that killed him. The framework honors him at the first rung of self-reference because the first rung of self-reference is exactly where he belongs, and because the framework's own consistency depends on the structure he formalized.

Remark 1.1 (Dependency, not tribute)**.** *The placement is a dependency ordering, not a symbolic gesture. Turing is at the rung that is structurally prerequisite to every later rung involving any form of self-reading, which is all of them. A framework that delivers its later content without honoring this level is a framework that silently assumes what Turing made explicit.*

2 What is new at R_4

The first structural level contributes the prime $P_1 = 2$ and the second contributes $P_2 = 3$. The third level contributes no new prime. It contributes the first integer whose factorization is a nontrivial *power* of an existing prime:

$$R_4 = P_1^2 = 4.$$

Definition 2.1 (Self-reference rung)**.** *The* first self-reference rung *is the smallest composite integer whose prime factorization is P^e for some prime P already introduced and some exponent $e \geq 2$. This rung is $R_4 = P_1^2$.*

Proposition 2.2 (R_4 is the first off-line point)**.** *The integer R_4 is the smallest positive integer whose prime-coordinate representation places it* strictly off *the number line of either axis introduced so far.*

Proof. In the prime-coordinate system, a positive integer n is represented by the tuple of exponents of its prime factorization. The integers $1, 2, 3$ have coordinates $(0, 0)$, $(1, 0)$, $(0, 1)$ respectively; these all lie on the two coordinate axes. The integer $4 = P_1^2$ has coordinates $(2, 0)$, which is the first point at height strictly greater than 1 along an existing axis. It is not on the interior of the plane yet (its second coordinate is still zero), but it is the first point whose position on the P_1-axis is not reachable by a single step of addition from the preceding point: the step from 2 to 4 doubles, while the step from 3 to 5 would add. □

Key idea

R_4 is the first integer whose construction uses the same prime twice. The second use of the same prime is the first act of self-reference in the framework.

3 Time first becomes possible

At the levels of P_1 and P_2 there is no before and no after. A configuration is either in state $+$ or state $-$ (first level), and it is at position (a, b) on the plane (second level). Neither level contains an ordering that would distinguish "previous" from "current."

At R_4 this changes.

Proposition 3.1 (First before-and-after)**.** *A self-referential rung supports a two-state ordering. The structure P_1^2 is read as "the first P_1 followed by the second P_1," and the two copies of P_1 are distinguishable by their position in the exponentiation.*

Proof. The exponent 2 in P_1^2 is not commutative in an ordering sense: the expression is read as a product of two factors, and the two factors occupy distinct positions in the product even though they have the same value. Assigning the first copy to "before" and the second copy to "after" produces a two-state ordering on the rung. This is the first rung on which such an ordering is possible, because every lower rung either has no repeated prime (trivial ordering) or no composite structure at all. □

Remark 3.2 (Time is possible, not yet stable)**.** *Time first becomes* possible *at R_4 because the structure now supports "one reading follows another." It does not become* stable *until $P_3 = 5$, the level of time proper (Bergson), where the third dimension is added and the framework acquires a duration coordinate. The two levels are distinct: R_4 makes time expressible, P_3 makes it live.*

4 Memory as the structural content of self-reference

A system that can read its own state is a system with memory. Memory is not added to the framework as a separate primitive; it is the structural content of self-reference at R_4.

Definition 4.1 (Memory at R_4)**.** *A configuration at R_4* has memory *in the sense that its factorization contains two distinguishable copies of P_1, each of which can be read independently while the other is held. The held copy is the memory content; the read copy is the current state.*

Proposition 4.2 (Smallest memory rung)**.** *R_4 is the smallest rung supporting memory in the sense of Definition 4.1.*

Proof. A rung R_n with $n < 4$ is either trivial or a single prime; neither contains two copies of anything. $R_4 = P_1^2$ contains exactly two copies of P_1 and is therefore the smallest rung on which one copy can be held while the other is read. □

Key idea

Memory is the arithmetic content of a squared prime. A single prime has no memory (there is nothing to hold while something else is read); a squared prime has exactly one bit of memory; higher powers have more.

5 Computation as the action of self-reference

Turing's contribution is not just the observation that self-reference is possible. It is the construction of a mechanical process that uses self-reference to compute. The framework records this at R_4 as the appearance of an operation whose input and output are both configurations of the same rung.

Proposition 5.1 (Computation at a self-reference rung)**.** *A rung R_n with a repeated prime factor admits a self-map $f : R_n \to R_n$ whose definition is allowed to refer to its own graph. This is not possible at rungs without a repeated prime factor, because any such rung has no structural slot in which to hold its own definition while the definition is being applied.*

Proof. The two copies of P_1 in R_4 provide two slots. One slot can hold the definition of f; the other can hold the argument to which f is being applied. A rung with a single copy of each prime has only one slot per prime, and any self-map would have to overwrite its own definition with its argument — which is not a self-map but an overwrite. □

Remark 5.2. *The present article does not develop the full theory of rung-indexed computation. That belongs to a later paper in the series and requires additional structure (composition, halting conditions, and the interaction of the self-map with the polarity matrix of rung R_{12}). What we claim here is only that R_4 is the smallest rung on which a self-map is structurally possible, and that this is the arithmetic content of the computation-as-self-reference insight.*

6 What is forced at this level

- The first composite rung is $R_4 = P_1^2$. No smaller composite exists.
- Self-reference becomes possible: the rung's structure contains two distinguishable copies of the same prime, and one copy can be read while the other is held.
- Time becomes expressible as a before/after ordering, though it does not yet live as a continuous coordinate.
- Memory becomes available as the held-while-read content of the rung's factorization.
- Computation, in the sense of a self-map applied to its own definition, first becomes structurally possible.

Takeaway

R_4 is not a new dimension; it is the first act of the framework doing something to itself. Every later rung that involves reading its own state — which is, eventually, all of them — traces its possibility back to this rung.

7 Summary

At the third structural level the rung $R_4 = P_1^2$ introduces self-reference as the first act of the framework doing something to itself. It is the smallest composite rung, the first integer whose factorization repeats a prime, and the first place at which before-and-after, memory, and computation become structurally possible. The level is centered on Turing, whose formalization of self-reference as computation is prerequisite to every later step of this series and to the existence of any machine that can read it.