The Polarity Matrix: Why There Are Exactly Four Arithmetic Operations

Pedro Henrique Correa Garcia
Joinville, Santa Catarina, Brazil
garcia.pedro.wow@gmail.com

April 2026

Abstract

The four elementary arithmetic operations — addition, subtraction, multiplication, division, and their higher forms exponentiation and logarithm — are traditionally presented as independent definitions. We show that at rung 12 of the prime ladder they are not independent: they are the four sign combinations of a single underlying operation N^N, and they are forced to be exactly four because a two-sided polarity matrix admits exactly four entries. The multiplicative form $12 = 2^2 \cdot 3$ — duality-squared times space — is the first rung at which all four combinations are simultaneously realizable, which is why rung 12 is the closure point of the arithmetic layer of the manifold.

1 Introduction

In Article 0 the ladder construction arrived at rung 12 with the claim that rung 12 is the "first configuration where all four arithmetic operations close." That claim was stated without proof. The present article supplies the proof, by constructing the polarity matrix $\mathcal{P}$ whose four entries are exactly the four operations, and showing that the entries are not chosen but counted.

The construction proceeds in three steps. First, we identify the two fundamental polarities that the manifold carries at rung 12. Second, we construct the 2×2 polarity matrix and read off its four entries. Third, we check that no fifth operation is possible, and that the four entries do not collapse into fewer.

Key idea

There are four arithmetic operations because there are two polarities and $2 \times 2 = 4$. The operations are not axiomatic — they are a count.

2 The two polarities

At every rung of the ladder the manifold carries a state equation $\psi = \kappa + x$. The two ingredients κ and x have different origins:

- κ is generator-sourced: it inherits its values from the Khronos direction, from the outside. Its polarity is *additive / subtractive*: κ can be positive (entering the state) or negative (leaving the state).

- x is manifold-coordinate-sourced: it inherits its values from the interior of the manifold. Its polarity is *multiplicative / divisive*: x can be multiplying the state (scaling up) or dividing it (scaling down).

These are the only two polarities the manifold carries at rung 12. We will not assume this; we will derive it.

Proposition 2.1 (Exactly two polarities at rung 12)**.** *At rung* $12 = 2^2 \cdot 3$*, the number of independent polarities is exactly* 2.

Proof. A polarity is a $\mathbb{Z}/2$-valued invariant of the state. The state is built from the arithmetic data of the factorization $12 = 2^2 \cdot 3$. The prime 2 contributes a parity (the sign of κ, which is how duality acts on additive structure). The prime 3 contributes a three-axis structure, which in dimension 1 (the reading relevant at rung 12) reduces to a scale direction (the sign of the exponent of x). The prime content of 12 is $\{2, 3\}$; each of them contributes exactly one polarity bit. Therefore the total number of polarities is 2. $\square$

Remark 2.2. *The proof relies on the prime-dimensional reading of the framework: each prime in the factorization of the rung number contributes its own degree of freedom, and at rung* 12 *the relevant primes are* 2 *and* 3*. Primes that do not appear in the factorization do not contribute polarities at that rung. This is why rung* $10 = 2 \cdot 5$ *and rung* $12 = 2^2 \cdot 3$ *carry different polarity structures even though both are two-prime composites.*

3 The polarity matrix

We now assemble the two polarities into a single 2×2 matrix. Let N^+N^+ denote the state with both polarities positive, N^+N^- denote positive-κ / negative-x, N^-N^+ denote negative-κ / positive-x, and N^-N^- denote both negative.

Definition 3.1 (Polarity matrix)**.** *The* polarity matrix *at rung* 12 *is*

$$\mathcal{P} \;=\; \begin{pmatrix} N^+N^+ & N^+N^- \\ N^-N^+ & N^-N^- \end{pmatrix}. \tag{1}$$

Each entry of $\mathcal{P}$ is a simultaneous choice of sign for the additive polarity of κ and the multiplicative polarity of x. The four entries are the four possible combinations.

4 Reading the entries as operations

Each entry of $\mathcal{P}$ corresponds to a way of combining two states. We now read them off.

Theorem 4.1 (Four operations from four polarities)**.** *The four entries of* $\mathcal{P}$ *correspond, in order, to the four operations:*

$$\mathcal{P} \;\longleftrightarrow\; \begin{pmatrix} \textbf{\textit{exponentiation}} & \textbf{\textit{multiplication}} \\ \textbf{\textit{division}} & \textbf{\textit{logarithm}} \end{pmatrix}.$$

Proof. We read each entry through the state equation $\psi = \kappa + x$, with κ carrying additive polarity and x carrying multiplicative polarity.

Entry N^+N^+ (both positive). The additive polarity is positive (κ enters), the multiplicative polarity is positive (x scales up). The operation that simultaneously adds structure and scales it up is exponentiation: $a^b = a \cdot a \cdots a$ (adding b copies of the scaling operation a). This is the canonical "positive-positive" combination.

Entry N^+N^- (positive κ, negative x). Additive polarity positive, multiplicative polarity negative: κ enters but x does not scale up. The operation that adds structure without scaling is multiplication: $a \cdot b$ combines a and b additively-in-logarithm but does not further exponentiate.

Entry N^-N^+ (negative κ, positive x). Additive polarity negative, multiplicative polarity positive: κ leaves but x scales. The operation that removes structure while preserving scale is division: a/b removes b's additive contribution to a's multiplicative structure.

Entry N^-N^- (both negative). κ leaves, x does not scale. The operation that simultaneously removes structure and removes scaling is logarithm: $\log_a(b)$ takes two things and returns the "how many copies" count, which is the maximally deconstructive of the four operations. □

Remark 4.2. *The traditional pairing (exp $\leftrightarrow$ log, $\times \leftrightarrow \div$) corresponds to the two diagonals of $\mathcal{P}$. Exp and log sit on the main diagonal; multiplication and division sit on the anti-diagonal. Inverses are always diagonal entries of the polarity matrix because inversion flips both polarities simultaneously.*

5 Closure and the absence of a fifth operation

A natural objection: why stop at four? What prevents the framework from generating a fifth or sixth operation beyond the canonical ones?

Theorem 5.1 (Closure of the arithmetic layer)**.** *There is no fifth elementary operation compatible with the polarity matrix $\mathcal{P}$ at rung* 12*.*

Proof. A fifth operation would require a fifth entry in $\mathcal{P}$, which would require either a third polarity or a non-binary valuation for one of the existing polarities. The third polarity is ruled out by Proposition 2.1: the factorization of 12 supplies exactly two primes, and each prime contributes one polarity bit. A non-binary valuation is ruled out because polarity in the manifold matrix is, by construction of the state equation $\psi = \kappa + x$, a $\mathbb{Z}/2$ invariant. Any operation beyond the four entries of $\mathcal{P}$ would have to live outside the arithmetic layer — for example, in the non-local / archival layer introduced in a later article — and would not be an arithmetic operation in the classical sense. □

Takeaway

The four arithmetic operations are the polarity matrix of rung 12. There is no fifth operation for the same reason there is no fifth entry in a 2×2 matrix.

6 Why rung 12 and not earlier

The polarity matrix requires two independent polarity bits. The earliest rung at which two independent polarities can simultaneously exist is the earliest rung whose factorization contains two distinct primes with the correct multiplicities.

- Rung $6 = 2 \cdot 3$ carries both primes, but with multiplicities 1 and 1. The additive polarity (from 2) exists, but without a 2^2 there is no way to form the positive-positive *diagonal* of $\mathcal{P}$: exponentiation requires "duality acting on itself," i.e. 2^2.

- Rung $8 = 2^3$ has 2^3 but no second prime: it cannot carry the multiplicative polarity at all. The decay hammer of Article 2 applies here.

- Rung $9 = 3^2$ has the second prime squared but no first prime: only half the matrix is present.

- Rung $10 = 2 \cdot 5$ has two primes but the wrong second prime: the prime 5 carries allocation structure (Article 2), not polarity structure, so it cannot contribute a polarity bit.

- Rung 11 is a single prime (the rotation prime of Article 0); no polarity matrix is possible there.

- Rung $12 = 2^2 \cdot 3$ is the first rung that has: 2^2 (duality squared, supporting the exponentiation entry), 3 (space, supporting the multiplicative polarity), and both simultaneously. It is the closure point of the arithmetic layer.

Proposition 6.1 (First complete polarity matrix)**.** *Rung $12 = 2^2 \cdot 3$ is the smallest rung whose factorization simultaneously supports the additive polarity bit (from 2^k with $k \geq 2$) and the multiplicative polarity bit (from a second prime $p \in \{3\}$ with p carrying space rather than allocation). No smaller rung has both.*

7 Closure at rung R_{12} as an arithmetic event

Rung R_{12} is the first rung of the prime ladder whose factorization contains both a prime square and a distinct second prime. The factorization $R_{12} = P_1^2 \cdot P_2$ is the smallest integer that supplies two independent polarity bits simultaneously: one from the squared prime, one from the second prime. No rung strictly below R_{12} carries both bits, and every rung strictly above R_{12} carries the two bits plus additional content that does not fit into the 2×2 polarity matrix.

The arithmetic content of this section is therefore a bare arithmetic fact about small integers: R_{12} is the first small integer whose factorization closes the polarity matrix, and it does so by exhibiting the exact pair of factor types (P_1^2, P_2) that the matrix requires. Any downstream interpretation of this closure at a particular rung is left to the reader's own domain.

Takeaway

$R_{12} = P_1^2 \cdot P_2$ is the first rung whose factorization supplies both a squared-prime bit and a distinct-prime bit, and therefore the first rung at which the four arithmetic operations close in a single polarity matrix. The closure is a theorem of rung arithmetic.

8 Summary

At rung 12 of the prime ladder, the factorization $12 = 2^2 \cdot 3$ supplies exactly two polarity bits: an additive bit from 2^2 and a multiplicative bit from 3. The four combinations of these bits form the 2×2 polarity matrix $\mathcal{P}$ of (1), and its four entries correspond, in order, to exponentiation, multiplication, division, and logarithm.

There are exactly four elementary arithmetic operations because the polarity matrix has exactly four entries. There is no fifth because the factorization of 12 supplies no third polarity, and a third polarity would be required to generate a fifth entry.

Rung 12 is the closure point of the arithmetic layer. Earlier rungs carry strict subsets of the polarity structure; later rungs add non-local polarities that live in the archival layer and are not arithmetic in the classical sense.