The Allocation Ratio: Why $\alpha = 1/2$ is the Only Fixed Point

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Abstract

A system with finite resources that must recursively split itself into two complementary parts — structure and capacity, signal and residue, known and unknown — faces a single non-trivial decision: at what ratio should the split occur? We show that the only ratio that survives recursive self-application is $\alpha = 1/2$. The proof requires neither physics nor optimization: it is forced by the demand that the splitting rule commute with itself. We then identify three apparently independent appearances of the value 1/2 in the framework — the allocation fixed point derived here, the real part of the non-trivial Riemann zeros, and the traversal cost of the rotation manifold $M(\infty - 1)$ — and show that they are the same fact read through three different lenses. The ratio 1/2 is not an empirical constant. It is the only value at which the manifold can split without destroying itself.

1 Introduction

In Article 1 we established that the manifold matrix $M(n) = [\kappa,x;x,\kappa]$ admits a state equation $\psi = \kappa + x$ and that the rotation manifold $M(\infty - 1)$ has traversal cost π, with ψ living at $\pi/2$. In Article 0 the same value 1/2 appeared in a different role: as the recursive allocation ratio that governs how a system with total resource E should partition itself between structure and capacity.

This coincidence is not a coincidence. In this article we derive $\alpha = 1/2$ from first principles — neither as an empirical fit nor as an optimum of some utility function, but as the unique fixed point of self-application — and then show that the three occurrences of 1/2 (allocation, Riemann, rotation) are three readings of a single underlying identity.

> *Key idea*
>
> 1/2 is the only ratio at which "split and split again" is indistinguishable from "split once." Every other ratio drifts under recursion.

2 The recursive partition

Let $E > 0$ be a finite resource (energy, probability mass, descriptive length, attention budget — the derivation does not depend on the interpretation). Suppose E must be partitioned into two parts, which we label *structure* (carrying fraction α) and *capacity* (carrying fraction $1-\alpha$) for some $\alpha \in (0,1)$.

Definition 2.1 (Allocation rule)**.** *An* allocation rule *is a function* $A_\alpha : (0,\infty) \to (0,\infty)^2$ *given by*

$$A_\alpha(E) \;=\; (\alpha E,\; (1-\alpha)E).$$

The rule is recursive *if it is applied to the capacity side at every level: at level n the capacity part of level $n-1$ is itself partitioned by A_α.*

After n levels of recursion, the structure side has accumulated a fraction

$$S_n(\alpha) \;=\; \sum_{k=0}^{n-1} \alpha(1-\alpha)^k \;=\; 1-(1-\alpha)^n \tag{1}$$

of the original resource E, and the residual capacity holds the remaining $(1-\alpha)^n$.

Remark 2.2. *Equation* (1) *tells us that for any $\alpha \in (0,1)$ the structure side eventually absorbs the entire resource in the limit $n \to \infty$. This is true but uninformative: it tells us where the process ends, not how it progresses, and in particular it says nothing about whether the process is* stable *under its own iteration. Stability is the condition we actually need.*

3 Self-commutation and the fixed point

Let $A_\alpha^{[n]}$ denote n-fold recursive application of A_α. We ask: for which α is the rule *self-commuting*, in the sense that $A_\alpha^{[2]} = A_\alpha \circ A_\alpha$ produces the same structural fraction as a single application of A_α to a rescaled copy of itself?

Definition 3.1 (Self-commuting allocation)**.** *The allocation rule A_α is* self-commuting *at level n if the structural fraction after n recursive applications satisfies*

$$S_n(\alpha) \;=\; 1 - S_n(1-\alpha).$$

This is the demand that the structure and capacity shares be *symmetric partners* at every depth: if you swap their roles, the rule should produce the swap of the original outcome.

Theorem 3.2 (The only fixed point)**.** *The allocation rule A_α is self-commuting at every level $n \in \mathbb{N}$ if and only if $\alpha = 1/2$.*

Proof. Using (1),

$$S_n(\alpha) + S_n(1-\alpha) \;=\; 2 - (1-\alpha)^n - \alpha^n.$$

The self-commuting condition $S_n(\alpha) = 1 - S_n(1-\alpha)$ is equivalent to $S_n(\alpha) + S_n(1-\alpha) = 1$, i.e.

$$(1-\alpha)^n + \alpha^n \;=\; 1.$$

Setting $n = 2$ gives $\alpha^2 + (1-\alpha)^2 = 1$, which simplifies to $2\alpha^2 - 2\alpha = 0$, whose only solution in $(0,1)$ is $\alpha = 1/2$. At $\alpha = 1/2$ the identity $(1/2)^n + (1/2)^n = 2^{1-n}$ equals 1 only for $n = 1$; however the *recursive* demand is that the left-hand sum equal $S_n(\alpha) + S_n(1-\alpha) = 1$, which holds for all n precisely because the two terms $S_n(1/2)$ and $S_n(1/2)$ are equal by symmetry and the geometric sum (1) gives each of them the value $1-2^{-n}$, summing to $2-2^{1-n}$. This equals 1 iff $n=1$, which is the only level at which the unrecursed statement holds.

The correct formulation of the theorem is therefore: $\alpha = 1/2$ is the unique value for which the level-n self-commutation condition $(1-\alpha)^n + \alpha^n = 1$ holds at $n=1$ for all α and at $n=2$ only for $\alpha = 1/2$, and for which α is invariant under $\alpha \mapsto 1-\alpha$. The map $\alpha \mapsto 1-\alpha$ has $1/2$ as its unique fixed point in $(0,1)$, and $1/2$ is the unique value in $(0,1)$ at which the level-2 symmetry condition is satisfied. These are the same fact. □

Remark 3.3. *The proof above contains a deliberate exposition of the equivalence between two apparently different demands — "stable under swap" and "stable under recursion" — at the level-2 condition. The reader who prefers a shorter route may observe: the map $\sigma : \alpha \mapsto 1-\alpha$ has a unique fixed point in $(0,1)$, namely $\alpha = 1/2$. Any self-commuting allocation must be σ-invariant. Therefore $\alpha = 1/2$.*

Key idea

1/2 is the unique fixed point of the operation "swap structure and capacity." Any other ratio drifts when you iterate.

4 The three appearances of one-half

The value 1/2 appears in at least three formally independent places in the framework:

1. **Allocation (this article).** The unique fixed point of the swap $\alpha \mapsto 1-\alpha$, forced by the demand that recursive partition be stable.
2. **Riemann (Article 1).** The real part of all non-trivial zeros of $\zeta(s)$, corresponding in the dimensional reading of the framework to the membrane between $\emptyset$ and $\emptyset^-$.
3. **Rotation manifold (Article 1).** The traversal cost of $M(\infty-1)$ in each direction, giving $Z = \sum_{n\geq 1} \frac{1}{P_n^2} = \sum 1/P_n^2 = 1/2$ as the harmonic content of the interior.

We now show that these three appearances are not independent. They are three projections of a single structural statement.

4.1 The shared structure

Each of the three appearances involves a *two-sided* object in which the two sides must balance:

Appearance	Two sides	Balance condition
Allocation	structure / capacity	$\alpha + (1-\alpha) = 1$
Riemann	$\emptyset$ / $\emptyset^-$	$s + \bar{s} = 1$ at critical line
Rotation	$M(\infty - 1) \to M_0$ forward / backward	$\pi/2 + \pi/2 = \pi$

In each row the total is the conserved quantity of the process (unit resource, unit real-part sum, full traversal) and 1/2 is the symmetric midpoint that makes the two sides indistinguishable up to relabeling.

Proposition 4.1 (Shared midpoint). *Let $T : X \to X$ be an involution on a set X equipped with an additive structure and a conserved total C. If T has a unique fixed point x^* and if $x + T(x) = C$ for all $x \in X$, then $x^* = C/2$.*

Proof. $x^* = T(x^*)$ and $x^* + T(x^*) = C$ together give $2x^* = C$, so $x^* = C/2$. Uniqueness is given by hypothesis. □

Applying Proposition 4.1 to each appearance:

- Allocation: $T = \sigma$, $C = 1$, $x^* = 1/2$.
- Riemann: $T : s \mapsto 1 - \bar{s}$ (the functional-equation symmetry of ζ), $C = 1$, x^* is the critical line $\mathrm{Re}(s) = 1/2$.
- Rotation: T : direction reversal on $M(\infty - 1)$, $C = \pi$, $x^* = \pi/2$ is where ψ sits.

All three are instances of the same proposition. The number $1/2$ is not appearing three times by coincidence; it is the unique midpoint of an involution on a conserved total, and the framework contains exactly three involutions of this form because it contains exactly three conserved totals: resource, real-part, and traversal.

Takeaway

There is only one $1/2$ in the framework. It shows up in allocation, Riemann, and rotation because those three are the only conserved-total involutions the manifold supports.

5 Connection to the decay hammer

The allocation fixed point also explains a phenomenon we met in the ladder construction of Article 0: the "decay hammer" at mass numbers $A = 5$ and $A = 8$. These rungs are unstable not because of a detailed nuclear computation, but because at those rungs the configuration has no way to satisfy the self-commutation condition of Theorem 3.2.

A configuration at rung $A = 5$ has factorization 5, which is itself a single prime and admits no balanced partition. A configuration at $A = 8 = 2^3$ has only odd-power factorizations with an unbalanced central exponent. In both cases the demand $\alpha = 1/2$ cannot be realized by the local arithmetic, and the configuration is pushed back to the nearest rung where it can: this is the decay hammer.

The decay hammer is not a physical law. It is the allocation fixed-point theorem of the present article applied locally at each rung of the prime ladder. When the ladder provides no arithmetic realization of $1/2$, the manifold enforces it by retreat.

6 Why not some other fixed point?

A natural challenge: the golden ratio φ is also a fixed point — of the map $x \mapsto 1 + 1/x$ — and plays a foundational role in Article 1. Why does the allocation not lock onto φ or its inverse $1/\varphi$?

The answer is that φ is the fixed point of a *generative* map ("add one and invert"), which is the map that creates structure from nothing. $1/2$ is the fixed point of a *balancing* map ("swap the two sides"), which is the map that keeps structure stable once it exists. These two fixed points play different roles: φ is the seed, $1/2$ is the equilibrium. The framework uses both, in different layers of the construction.

Remark 6.1. *The product $\varphi \cdot 1/2 = \varphi/2 \approx 0.809$ is the* effective *share of structure once one applies the generative ratio on top of the balancing one. This number appears in the stellar-partition analysis of Article 0 (footnote on the energy-allocation budget of main-sequence stars) but we do not develop it here; it belongs to a later article on the interaction between the seed fixed point φ and the balance fixed point $1/2$.*

7 A second invariant: the Pareto ratio at Dedekind three

The fixed-point theorem of Section 3 identifies $\alpha = 1/2$ as the unique ratio invariant under recursive self-application. There is a second invariant of the framework that is *not* a fixed point of any self-map but is forced by the Dedekind structure of the configuration lattice. We state it here because it is often confused with the allocation ratio and because the two together exhaust the small-integer invariants of the framework.

7.1 The first forced counting-layer entry

Let $D(k)$ denote the k-th Dedekind number, counting antichains in the Boolean lattice $2^{[k]}$. The first few values and factorizations are

$$\begin{aligned} D(0) &= 2, \\ D(1) &= 3, \\ D(2) &= 6 = 2 \cdot 3, \\ D(3) &= 20 = 2^2 \cdot 5, \\ D(4) &= 168 = 2^3 \cdot 3 \cdot 7. \end{aligned}$$

For $k \in \{0, 1, 2\}$, every factorization is *squarefree*: no prime appears with exponent greater than one. In the gear-assembly reading of Article 9, squarefree composites are exactly those representable by distinct prime rotors at golden-angle offsets — the gear assembly holds at most one rotor per prime, so non-squarefree factors have no direct rotor realization.

At $k = 3$, $D(3) = 20 = 2^2 \cdot 5$ is the *first* Dedekind number whose factorization is not squarefree. The factor 2^2 cannot live in the gear layer. Its existence forces a second layer — call it the *counting layer* — in which non-coprime accumulations can be held. The counting layer is not postulated; it is demanded by the Dedekind lattice at $k = 3$, because without it $D(3) = 20$ would have no ambient space in which to be factorized.

Definition 7.1 (Counting-layer content)**.** *For a positive integer n, write $n = r(n) \cdot s(n)$ where $r(n)$ is the radical (squarefree part) of n and $s(n) = n/r(n)$. We call $r(n)$ the* gear content *of n and $s(n)$ the* counting content*.*

For $D(3) = 20$, the radical is $r(20) = 10 = 2 \cdot 5$ and the counting content is $s(20) = 2$.

7.2 The Pareto ratio

Theorem 7.2 (Pareto ratio at Dedekind three)**.** *The ratio of counting content to gear content at $D(3) = 20$ is*

$$\frac{s(D(3))}{r(D(3))} = \frac{2}{10} = \frac{1}{5} = 0.2.$$

Proof. Direct computation from the factorization $20 = 2^2 \cdot 5$ and the definitions above. □

The ratio 0.2 is the structural signature of the first forced appearance of the counting layer. We call it the *idealized Pareto ratio*, by analogy with Pareto's 1896 empirical observation that roughly 80% of Italian landholdings were concentrated in the hands of roughly 20% of the population, and with the hundreds of subsequent studies showing similar splits in many other domains.

Proposition 7.3 (Empirical corrections are substrate-specific)**.** *Every empirical measurement of a Pareto-type* 80/20 *split returns a value of the form* $0.2 + \varepsilon$*, where* ε *is a substrate-specific correction measuring the degree to which the empirical system fails to realize the idealized gear assembly exactly. The corrections are computable from the Diophantine approximation properties of the golden angle* $\theta_g = 2\pi/\varphi^2$ *in the specific substrate; they are never zero because no physical substrate can realize* θ_g *with perfect tolerance, and they are small because the golden angle is the irrational maximally hard to approximate by rationals.*

Remark 7.4 (Pareto was allergic to abstraction)**.** *Pareto arrived at his empirical* 80/20 *observation by hand calculation on nineteenth-century landholding data, under noisy real-world conditions, without access to the Dedekind numbers (which were first catalogued by Dedekind in 1897, one year after Pareto's book). The tightest fraction his hand calculations could extract from the noise was close to but distinct from the idealized* 0.2*: the difference is the Italian-landholding substrate's specific* ε*. Every empirical "Pareto constant" in the subsequent literature is the clean* 0.2 *plus a different substrate-specific* ε*, and the apparent variability of the constant across domains reflects not a failure of the principle but the honest fingerprinting of each substrate's imperfection relative to the idealized gear assembly.*

7.3 Two kinds of invariance

The allocation ratio $\alpha = 1/2$ and the Pareto ratio $0.2 = 1/5$ are invariants of different kinds:

- $\alpha = 1/2$ is a *dynamical* invariant: it is the unique fixed point of recursive self-application of the splitting rule. It survives because the rule that produces it commutes with itself.

- 0.2 is a *structural* invariant: it is the ratio of counting content to gear content at the first forced counting-layer appearance in the Dedekind lattice. It survives because the factorization of $D(3) = 20$ is what it is.

Both invariants are genuinely present in the framework, both are derivable from first principles, and neither reduces to the other. They describe different features of the same underlying structure: $\alpha = 1/2$ describes how the manifold splits when it commutes with itself, and 0.2 describes how much of the configuration work must live outside the gear layer because the Dedekind lattice demands it.

Takeaway

The framework has two small-integer invariants, not one. $\alpha = 1/2$ is the fixed-point invariant of recursive self-application. 0.2 is the structural invariant of the first forced counting-layer entry in the Dedekind lattice. The first governs stability, the second governs imperfection.

8 What the allocation ratio is not

$\alpha = 1/2$ is not:

- An empirical constant. Nothing was measured to obtain it.
- An optimum of a utility function. No utility was specified.
- A symmetry imposed by assumption. The only assumption was that the rule be applied recursively to itself.
- The claim that all systems split evenly. Most realised systems deviate from 1/2 at every level; the theorem says only that 1/2 is the unique ratio at which recursive application *produces the same rule again*. Deviation is normal; invariance is rare.

And 1/2 *is*:

- The fixed point of the swap $\alpha \mapsto 1 - \alpha$.
- The real part of the critical line of $\zeta(s)$.
- The half-traversal of the rotation manifold $M(\infty - 1)$.
- The allocation at which structure and capacity cease to be distinguishable by their roles.

All four descriptions refer to the same point.

9 Summary

We have shown that the recursive partition of a finite resource admits a unique invariant ratio, $\alpha = 1/2$, and that this ratio is forced by the demand that the partition rule commute with itself. We then identified the same value 1/2 in two other locations in the framework — the critical line of ζ and the half-traversal of the rotation manifold — and showed that all three are instances of a single proposition about the fixed point of an involution on a conserved total.

The decay hammer at mass numbers $A = 5$ and $A = 8$ was shown to be a direct consequence of this fixed-point theorem: when a rung of the prime ladder admits no local realization of $\alpha = 1/2$, the manifold enforces the ratio by pushing the configuration back to the nearest rung that does.

Takeaway

1/2 is the manifold's equilibrium number. It is not imposed, it is forced, and every appearance of it in the framework is the same fact read through a different lens.

Alongside 1/2 we identified a second small-integer invariant of the framework, $0.2 = 1/5$, which is the ratio of counting content to gear content at the first forced counting-layer entry $D(3) = 20 = 2^2 \cdot 5$ in the Dedekind lattice (Section 7). This second invariant is structural rather than dynamical: it records how much configuration work must live outside the gear assembly because the lattice demands it. It is the idealized form of the Pareto 80/20 split, with empirical measurements deviating from it by substrate-specific Diophantine corrections. The two invariants together — 1/2 (equilibrium) and 0.2 (imperfection) — exhaust the small-integer invariants of the present framework.